Parametric phenomena of the particle dynamics in a periodic gravitational wave field

Alexander B. Balakin*, Veronika R. Kurbanova†
Department of General Relativity and Gravitation
Kazan State University, 420008 Kazan, Russia
and
Winfried Zimdahl‡
Fachbereich Physik, Universität Konstanz
D-78457 Konstanz, Germany
and
Institut für Theoretische Physik, Universität zu Köln
D-50937 Köln, Germany

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Abstract

We establish exactly solvable models for the motion of neutral particles, electrically charged point and spin particles (U(1) symmetry), isospin particles (SU(2) symmetry), and particles with color charges (SU(3) symmetry) in a gravitational wave background. Special attention is devoted to parametric effects induced by the gravitational field. In particular, we discuss parametric instabilities of the particle motion and parametric oscillations of the vectors of spin, isospin, and color charge.

1 Introduction

Periodic external fields are known to induce parametric phenomena in physical systems. This includes such effects as parametric oscillations (the oscillation

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*Electronic address: Alexander.Balakin@ksu.ru
†Electronic address: Veronika.Kurbanova@ksu.ru
‡Electronic address: zimdahl@thp.uni-koeln.de

frequency becomes a periodic function of time) and parametric instabilities (exponential growth of certain dynamical quantities) [1, 2, 3]. Classical examples are the parametric resonance in vibrations of mechanical and electrical systems [1, 2, 3] and plasma instabilities in external electromagnetic fields [4, 5]. From a mathematical point of view, such phenomena are described by differential equations with periodic coefficients which are subject to Floquet's theory [2, 3, 6, 7, 8, 9]. As we shall demonstrate here, equations of this type naturally appear if one considers the motion of different kinds of particles in a gravitational wave (GW) background. The gravitational wave may play the role of an external periodic pumping field. In fact, most of the attempts for a direct detection of gravitational waves rely on this concept. The mathematical similarity to dynamical equations which are known to describe parametric oscillations and resonances then naturally suggests the possibility of gravitationally induced parametric effects.

The idea of parametric phenomena in GW fields is not new. For a linearized GW field it was discussed in [10]. The first exactly solvable model for the evolution of a kinetic system in a *nonlinear* GW field, demonstrating explicitly the possibility of parametric excitation of a relativistic plasma by a periodic GW, has been established in [11]. At the same time, in the 90th, the problem of parametric resonance during the reheating phase of inflationary models has become an intensely elaborated topic in a cosmological context (see, e.g., [12, 13, 14, 15, 16]). More recently, investigations, concerning parametric phenomena in a GW field, have attracted attention again (see, e.g., [17, 18, 19]).

The purpose of the present paper is to clarify characteristic features of gravitationally induced parametric effects for simple dynamical configurations. As a first example we consider the motion of an electrically charged point particle which is simultaneously exposed to a constant magnetic field and a gravitational wave with front plane orthogonal to the magnetic field. In the second example we include an additional spin degree of freedom which is described by the Bargmann-Michel-Telegdi (BMT) equations [20]. In the third case the electrically charged spin-particle is replaced by a particle with isospin and the magnetic field is replaced by a corresponding Yang-Mills field. The isospin dynamics is governed by Wong's equation [21] for the three-dimensional isospin vector. Finally, we consider the motion of particles with color charge, described by Wong's equation for the eight color degrees of freedom. Using suitably specified Yang-Mills fields, we establish a general scheme which allows a unified treatment of the particle dynamics for all four cases. We show that on a periodic GW background this dynamics is characterized by Hill and Mathieu equations. Well-known stability properties of the latter allow us to classify the particle motion accordingly. This implies parametric oscillations and/or parametric instabilities as generic phenomena. The precession dynamics of the vectors of spin, isospin and color charge is coupled to the particle motion and parametrically driven as well.

The paper is organized as follows. In Section 2 we establish the basic dynamic equations for the Abelian and non-Abelian subcases to be discussed in the following. In Section 3 the Yang-Mills fields for the latter cases are specified. The (exact) gravitational background is characterized in Section 4. Section 5

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is devoted to a compact, general solution for the particle dynamics. A "sandwich" GW is considered as a special case. The spin precession for an electrically charged particle is the subject of Section 6. Sections 7 and 8 discuss the dynamics in the spaces of isospin and color charge, respectively. In Section 9 we summarize our main results. We use units in which $\hbar = c = 1$.

2 Particle dynamics: Basic equations

Let us consider the evolution of relativistic point particles with either an electric charge and a spin-vector, or an isospin, or a color charge. The concepts of classical particles with isospin (for the SU(2) symmetry) or color charge (for the SU(3) symmetry) are generalizations of the electrically charged particles to the non-Abelian case (see, e.g., [22]). The dynamical equations for the particle momentum p^i, for the spin-vector S^i and for the charge $Q^{(A)}$, where (A) is a group index, are

$$\frac{\mathrm{D}p^i}{\mathrm{D}\tau} = \mathcal{F}^i \ , \quad \frac{\mathrm{D}S^i}{\mathrm{D}\tau} = \mathcal{G}^i \quad \text{and} \quad \frac{\mathrm{D}Q^{(A)}}{\mathrm{D}\tau} = \mathcal{G}^{(A)} \ , \tag{1}$$

respectively. Here, D denotes the covariant differential, τ is a parameter along the particle worldline and $\mathcal{F}^i$ is the force four vector which is orthogonal to the particle momentum $p^i = m\mathrm{d}x^i/\mathrm{d}\tau$, i.e., $p_i\mathcal{F}^i = 0$. The quantity $\mathcal{G}^i$ describes the spin rotation within the BMT theory. The quantity $\mathcal{G}^{(A)}$ is a vector in the group space which determines the non-Abelian charge evolution and plays a similar role for the charge as $\mathcal{F}^i$ plays for the momentum. The limiting case of neutral particles is characterized by $\mathcal{F}^i = \mathcal{G}^i = \mathcal{G}^{(A)} = 0$.

2.1 Electrically charged point particles

In this case the relevant force is the Lorentz force

$$\mathcal{F}^i = \frac{e}{m} F^i_{\cdot k} \, p^k \tag{2}$$

with

$$F_{ik} = \nabla_i A_k - \nabla_k A_i, \quad \nabla_k F^{ik} = 0 \ , \tag{3}$$

where F_{ik} is the Maxwell tensor. The particle under consideration is regarded here as a test particle. Moreover, the charge is constant which renders the third equation in (1) irrelevant.

2.2 Electrically charged spin particles

According to [20] the evolution of classical relativistic spin particles is governed by the equations

$$\frac{\mathrm{D}p^i}{\mathrm{D}\tau} = \frac{e}{m} F^i_{\cdot k} \, p^k \tag{4}$$

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and

$$\frac{\mathrm{D}S^i}{\mathrm{D}\tau} = \frac{e}{2m}\left[gF^i_{.k}\, S^k + \frac{(g-2)}{m^2}p^i F_{kl}S^k p^l \right] \ .$$

(5)

Here, S^i is the spin four-vector and g is the gyromagnetic ratio. Equation (5) describes the precession of the magnetic moment. It generalizes earlier non-relativistic equations by Thomas [23] and Bloch [24], which rely on the circumstance that the "expectation value of the vector operator representing the "spin" will necessarily follow the same time dependence as one would obtain from a classical equation of motion" (cf. [20]). While the particle momentum according to Eq. (4) is independent of the spin vector, the dynamics of the latter is coupled (at least via the covariant derivative) to the particle motion.

2.3 Isospin particles

Here we have a triplet $I^{(A)}$ of scalar fields representing a vector in the three-dimensional isospin space, i.e., $(A) = (1), (2), (3)$. This space has an euclidean metric $G_{(A)(B)}$. The relevant force

$$\mathcal{F}^i = \frac{g}{m}F^{(A)i}_{\quad \cdot\ k}\, p^k I^{(B)}G_{(A)(B)} \ ,$$

(6)

where g is the interaction constant, has been obtained by Kerner [25] and Wong [21]. The isospin dynamics is determined by Wong's equation [21]

$$\frac{\mathrm{D}}{\mathrm{D}\tau}I^{(A)} = -\frac{g}{m}\varepsilon^{(A)}_{\ \cdot\ (B)(C)}A^{(B)}_i p^i\, I^{(C)} \ ,$$

(7)

where we have used that the structure constants for the $SU(2)$ group coincide with the three-dimensional Levi-Civita symbol $\varepsilon^{(A)}_{\ \cdot\ (B)(C)}$. The quantities $A^{(A)}_i$ are the vector potentials in terms of which the Yang-Mills field strength tensor $F^{(B)}_{jk}$ is given by

$$F^{(B)}_{jk} = \nabla_j A^{(B)}_k - \nabla_k A^{(B)}_j + g\varepsilon^{(B)}_{\ \cdot\ (K)(L)}A^{(K)}_j A^{(L)}_k \ .$$

(8)

The Yang-Mills field equations are

$$g^{ij}\left[\nabla_i F^{(A)}_{jk} + g\varepsilon^{(A)}_{\ \cdot\ (B)(C)}A^{(B)}_i F^{(C)}_{jk} \right] = 0 \ .$$

(9)

Again we consider the particle motion in a given external field. Wong's equation represents a non-Abelian generalization of the equation of motion for electrically charged point particles. It can be obtained as the classical limit from quantum field theory for the case of sufficiently localized quantum states of the matter fields with characteristic length scales much smaller than those associated to the Yang-Mills fields [26, 22].

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2.4 Particles with color charge

For test particles with color charge the force $\mathcal{F}^i$ is given by

$$\mathcal{F}^i = \frac{g}{m} F^{(A)i}_{\cdot\; k} \, p^k Q^{(B)} G_{(A)(B)} \; , \tag{10}$$

with the field strength tensor

$$F^{(B)}_{jk} = \nabla_j A^{(B)}_k - \nabla_k A^{(B)}_j + g f^{(B)}_{\cdot\,(K)(L)} A^{(K)}_j A^{(L)}_k \; , \tag{11}$$

where the $f^{(B)}_{\cdot\,(K)(L)}$ are the structure constants of the SU(3)group. The field equations are

$$g^{ij} \left[\nabla_i F^{(A)}_{jk} + g f^{(A)}_{\cdot\,(B)(C)} A^{(B)}_i F^{(C)}_{jk} \right] = 0 \; . \tag{12}$$

The quantity $Q^{(A)}$ is the color charge with $(A) = (1) \ldots (8)$. Wong's equation in this case reads

$$\frac{\mathrm{D}}{\mathrm{D}\tau} Q^{(A)} = -\frac{g}{m} f^{(A)}_{\cdot\,(B)(C)} A^{(B)}_i p^i \, Q^{(C)} \; . \tag{13}$$

The structure constants $f^{(A)}_{\cdot\,(B)(C)}$ are characterized by the commutator relations

$$\left[\lambda_{(A)}, \lambda_{(B)} \right] = 2i f_{(A)(B)(C)} \lambda^{(C)} \; , \tag{14}$$

where $\lambda_{(A)}$ are the traceless, hermitian Gell-Mann matrices (see, e.g., [27, 28]). In detail we have

$$f_{(1)(2)(3)} = 1 \; , \qquad f_{(4)(5)(8)} = f_{(6)(7)(8)} = \frac{\sqrt{3}}{2} \; ,$$

$$f_{(1)(4)(7)} = f_{(2)(4)(6)} = f_{(2)(5)(7)} \quad = \quad f_{(3)(4)(5)} = -f_{(3)(6)(7)} = -f_{(1)(5)(6)} = \frac{1}{2} \; . \tag{15}$$

Below we shall also use the completely symmetric coefficients $d_{(A)(B)(C)}$ of the basic representation which are given by the anti-commutation relations

$$\left\{ \lambda_{(A)}, \lambda_{(B)} \right\} = \frac{4}{3} \delta_{(A)(B)} + 2 d_{(A)(B)(C)} \lambda^{(C)} \; , \tag{16}$$

where [29, 22]

$$d_{(1)(4)(6)} = d_{(1)(5)(7)} = d_{(2)(5)(6)} = d_{(3)(4)(4)} = d_{(3)(5)(5)} =$$

$$= -d_{(2)(4)(7)} = -d_{(3)(6)(6)} = -d_{(3)(7)(7)} = \frac{1}{2} \; ,$$

$$d_{(1)(1)(8)} = d_{(2)(2)(8)} = d_{(3)(3)(8)} = -d_{(8)(8)(8)} = -2d_{(4)(4)(8)} =$$

$$- 2d_{(5)(5)(8)} = -2d_{(6)(6)(8)} = -2d_{(7)(7)(8)} = \frac{1}{\sqrt{3}} \; . \tag{17}$$

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3 Yang-Mills fields with "parallel" potentials

It is known that for each solution of the general relativistic source free Maxwell equations one can construct a set of solutions of the general relativistic Yang-Mills equations [30]. Following Gal'tsov [31] we will refer to the corresponding Yang-Mills potentials as "parallel" potentials. The latter are characterized by

$$A_i^{(B)} = q^{(B)} A_i \ , \quad F_{ik}^{(A)} = q^{(A)} F_{ik} \ , \quad q^{(B)} q_{(B)} = 1 \ , \quad q^{(B)} = \text{const} \ . \tag{18}$$

Due to the antisymmetry of the structure coefficients the relations (8) and (11) as well as the equations (9) and (12) reduce to the linear Maxwell-type forms (3). Nevertheless, compared with Maxwell's theory there exists an additional degree of freedom, namely the direction of the vector $q^{(A)}$ in the group space [30]. Additionally, the structure coefficients $f^{(B)}_{\cdot (K)(L)}$ are different from zero which will result in a qualitatively different dynamics.

3.1 Isospin particles

In this case the ansatz (18) transforms the first of equations (1) with (6) into

$$\frac{\mathrm{D} p^i}{\mathrm{D} \tau} = \frac{g I^{(A)} q_{(A)}}{m} F^i_{\cdot \ k} \, p^k \ , \tag{19}$$

which has the structure of the equations of motion of a particle with charge $e \equiv g I^{(A)} q_{(A)}$ under the influence of the Lorentz force (2). Analogously, one can rewrite equation (7) for the isospin evolution,

$$\frac{\mathrm{d}}{\mathrm{d} \tau} I^{(A)} = -\Omega \ \varepsilon^{(A)}_{\cdot (B)(C)} q^{(B)} I^{(C)} \ , \quad \Omega \equiv \frac{g}{m} A_i p^i \ . \tag{20}$$

Because of the antisymmetry of the Levi-Civita symbols the equations (20) admit a quadratic integral of motion $I^{(A)} I_{(A)} = \text{const}$, which is a Casimir invariant [22], normalizable to $I^{(A)} I_{(A)} = 1$. In addition, we obtain from (20)

$$q_{(A)} \frac{\mathrm{d} I^{(A)}}{\mathrm{d} \tau} \equiv 0 \ , \quad \rightarrow \quad I^{(A)} q_{(A)} \equiv I = \text{const} \ . \tag{21}$$

Using the standard definition

$$\left[\vec{I}, \ \vec{\Omega} \right]^{(A)} \equiv \varepsilon^{(A)}_{\cdot (B)(C)} I^{(B)} \Omega^{(C)} \ , \quad \Omega^{(C)} \equiv \Omega \ q^{(C)} \ , \tag{22}$$

of the vector product, Eq. (20) may be written as an equation for the precession of $\vec{I}$,

$$\frac{\mathrm{d}}{\mathrm{d} \tau} \vec{I} = \left[\vec{I}, \ \vec{\Omega} \right] \ . \tag{23}$$

The "longitudinal" component $I^{(A)} q_{(A)}$, the projection of the dynamical variable $I^{(A)}$ on the "rotation axis" $q^{(A)}$, remains constant according to (21).

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3.2 Colored particles

Similar to the previous isospin case the color charge evolution equation (13) admits the existence of a quadratic integral of motion $Q^{(A)}Q_{(A)} = $ const which is the first Casimir invariant [22]. The condition (18) of parallelism in the color space provides a second integral of motion $Q^{(A)}q_{(A)} = $ const, as well. However, the corresponding 8-dimensional evolution equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Q^{(A)} = -\Omega H^{(A)}_{\cdot\,(C)}Q^{(C)} \ , \quad \Omega \equiv \frac{g}{m}A_i p^i \ , \quad H^{(A)}_{\cdot\,(C)} \equiv f^{(A)}_{\cdot\,(B)(C)}q^{(B)} \ , \quad (24)$$

are more complicated than the equation (23) for the isospin precession. Different from the $SU(2)$ model there exists a second Casimir invariant

$$\mathcal{Q} = d_{(A)(B)(C)}Q^{(A)}Q^{(B)}Q^{(C)} \ , \tag{25}$$

where $d_{(A)(B)(C)}$ are the totally symmetric group coefficients (17) of the given basic representation of the $SU(3)$ group. In detail it reads

$$\mathcal{Q} = -\frac{1}{\sqrt{3}}(Q^{(8)})^3 + \sqrt{3}Q^{(8)}\left[(Q^{(1)})^2 + (Q^{(2)})^2 + (Q^{(3)})^2\right] -$$

$$-\frac{\sqrt{3}}{2}Q^{(8)}\left[(Q^{(4)})^2 + (Q^{(5)})^2 + (Q^{(6)})^2 + (Q^{(7)})^2\right] + 3Q^{(1)}\left[Q^{(4)}Q^{(6)} + Q^{(5)}Q^{(7)}\right] +$$

$$+3Q^{(2)}\left[-Q^{(4)}Q^{(7)} + Q^{(5)}Q^{(6)}\right] + \frac{3}{2}Q^{(3)}\left[(Q^{(4)})^2 + (Q^{(5)})^2 - (Q^{(6)})^2 - (Q^{(7)})^2\right] \ . \tag{26}$$

4 Gravitational wave background

Our aim in this paper is to study the general dynamics outlined so far in the field of a plane-fronted GW with parallel rays (PP-wave). We assume the latter to be an exact solution of Einstein's vacuum field equations with a five-parametric group of isometries G_5, including a covariantly constant null Killing vector (KV) [32]. Gravitational waves are usually described either in Fermi coordinates or in the transverse-traceless (TT)-gauge. For the merits of each of these choices and for issues of gauge-invariance in the linearized theory see, e.g., [33]. In order to establish a comprehensive picture we start by sketching our basic setting for both cases. For computational ease most of the analysis will then be done in TT-coordinates.

4.1 PP-wave in Fermi coordinates

The corresponding line element

$$\mathrm{d}s^2 = 2\mathrm{d}\bar{u}\mathrm{d}\bar{v} - \mathrm{d}y^2 - \mathrm{d}z^2 - 2\mathcal{H}(\bar{u},y,z)\mathrm{d}\bar{u}^2 \ , \quad \bar{u} = \frac{t-x}{\sqrt{2}} \ , \quad \bar{v} = \frac{t+x}{\sqrt{2}} \ , \quad (27)$$

contains a harmonic function $\mathcal{H}$, obeying

$$\frac{\partial^2 \mathcal{H}}{\partial y^2} + \frac{\partial^2 \mathcal{H}}{\partial z^2} = 0 \ , \tag{28}$$

which is quadratic in y and z for a G_5 symmetry group. Explicitly, we have [32],

$$2\mathcal{H}(\bar{u}, y, z) = A(\bar{u}) \left(y^2 - z^2\right) + 2B(\bar{u}) \ yz \ , \tag{29}$$

where $A(\bar{u})$ and $B(\bar{u})$ are arbitrary functions of the retarded time $\bar{u}$. We may define a periodic GW by assuming the variables A and B to be periodic functions of $\bar{u}$. All non-vanishing components of the Riemann tensor, $R_{z\bar{u}z\bar{u}} = -R_{y\bar{u}y\bar{u}} = A(\bar{u})$ and $R_{y\bar{u}z\bar{u}} = -B(\bar{u})$, are periodic for this case. The GW metric in Fermi coordinates is non-singular for arbitrary retarded times because $\det(g_{ik}) \equiv -1 \neq 0$. However, the weak field approximation $\max|g_{ik}| << 1$ is correct only close to $y = 0$, $z = 0$.

4.2 PP-wave in TT-gauge

The line element in TT-gauge has the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - L^2 \left[\cosh 2\gamma \left(e^{2\beta}(\mathrm{d}x^2)^2 + e^{-2\beta}(\mathrm{d}x^3)^2\right) + 2\sinh 2\gamma \mathrm{d}x^2 dx^3\right] \ , \tag{30}$$

where $u = \left(t - x^1\right)/\sqrt{2}$ and $v = \left(t + x^1\right)/\sqrt{2}$ are the retarded and the advanced times, respectively. For this metric the three KVs which form an Abelian subgroup of G_5 are

$$\xi^i_{(v)} = \delta^i_v, \quad \xi^i_{(2)} = \delta^i_2, \quad \xi^i_{(3)} = \delta^i_3 \ . \tag{31}$$

The KV $\xi^i_{(v)}$ is a covariantly constant null vector, orthogonal to $\xi^i_{(2)}$ and $\xi^i_{(3)}$. The functions $\beta(u)$ and $\gamma(u)$ are arbitrary. We shall focus here on the case of periodic functions $\beta(u)$ and $\gamma(u)$. This definition of periodicity does not, in general, coincide with the definition in Fermi coordinates given above. However, both concepts of a periodic GW have the same weak-field limit, which in the TT-gauge is characterized by $L = 1$, $|\beta(u)| << 1$ and $|\gamma(u)| << 1$. Generally, the function $L(u)$ satisfies the Einstein equation [33]:

$$\ddot{L} + L \left((\dot{\beta})^2 \cosh^2 2\gamma + (\dot{\gamma})^2\right) = 0 \ , \tag{32}$$

where a dot denotes the derivative with respect to the retarded time u. We assume the hypersurface $u = 0$ to be the leading front of the GW with

$$\beta(0) = \gamma(0) = 0 \ , \quad L(0) = 1 \ , \quad \dot{\beta}(0) = \dot{\gamma}(0) = \dot{L}(0) = 0 \ . \tag{33}$$

For the special case $\gamma(u) \equiv 0$ (equivalent to $B(\bar{u}) \equiv 0$), corresponding to only one polarization direction, the transformation relations between Fermi and TT coordinates are

$$\begin{aligned}
\bar{u} &= u, \quad \bar{v} = v + \frac{1}{4}\left[(x^2)^2(L^2 e^{2\beta})^{\cdot} + (x^3)^2(L^2 e^{-2\beta})^{\cdot}\right] \ , \\
y &= Le^{\beta} \cdot x^2, \quad z = Le^{-\beta} \cdot x^3 \ , \quad A(u) = \ddot{\beta} + 2\dot{\beta}\frac{\dot{L}}{L} \ .
\end{aligned} \tag{34}$$

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The last formula clarifies the relation between the different periodicity definitions given above. For the physically relevant situation where the background factor L changes only slowly compared with the change of the wave factor β [cf. Ref. [33]] the last term in the formula for $A(u)$ in Eq. (34) can be neglected. Then we have $A(u) = \ddot{\beta}$ and both periodicity definitions coincide.

5   Particle dynamics: Solutions

We are interested here in the particle dynamics in given external gravitational and Yang-Mills fields. The restriction (18) to Yang-Mills fields with "parallel potentials" simplifies the Kerner-Wong equations since the quantities $I^{(A)}q_{(A)}$ and $Q^{(A)}q_{(A)}$ remain constant. We introduce the cumulative symbol σ for either e, or $gI^{(A)}q_{(A)}$, or $gQ^{(A)}q_{(A)}$, which allows us to write the equation of motion with either (2), or (6), or (10) in the unified form

$$\frac{\mathrm{D}p^i}{\mathrm{D}\tau} = \frac{\sigma}{m} F^i_{.\;k}\, p^k, \quad \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{p^i}{m} \; . \tag{35}$$

The orthogonality of the force to the particle momentum corresponds to the quadratic integral

$$g^{ik} p_i p_k = m^2 \; . \tag{36}$$

One may solve this relation for one of the components of the momentum. In Fermi coordinates we have

$$p_{\bar u} = \frac{1}{2 p_{\bar v}} \left[m^2 + p_y^2 + p_z^2 - 2\mathcal{H}(\bar u, y, z)\, p_{\bar v}^2 \right] \tag{37}$$

The analogous formula in TT coordinates is

$$p_u = \frac{1}{2 p_v} \left[m^2 - g^{\alpha\beta}\left(u \right) p_\alpha p_\beta \right] \; , \tag{38}$$

where greek indices run from 2 to 3. Both in Fermi and in TT coordinates the covariantly constant null KV has the form $\xi^i_{(\bar v)} = \delta^i_{\bar v}$ and $\xi^i_{(v)} = \delta^i_v$, respectively. Below we shall restrict ourselves to fields which satisfy

$$\xi^i_{(\bar v)} F_{ik} = \xi^i_{(v)} F_{ik} = 0 \; . \tag{39}$$

This implies that the quantities $\xi^i_{(\bar v)} p_i$ and $\xi^i_{(v)} p_i$ are integrals of motion (see e.g. [33]),

$$\xi^i_{(\bar v)} p_i = \xi^i_{(v)} p_i = p_v = C_v = \mathrm{const} \; . \tag{40}$$

Using the general relationship

$$m \frac{du}{d\tau} = p^u = p^{\bar u} = p_v = C_v \; , \tag{41}$$

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one can reparametrize the remaining equations for $C_v \neq 0$, by means of the linear formula (notice that $\bar{u} = u$ [cf. Eq. (34)] holds also in the general case)

$$\tau = \tau_0 + \frac{m}{C_v} u \ . \tag{42}$$

We start our solution procedure by first recalling the reference case of neutral particles.

5.1 Neutral particles

For a vanishing generalized charge σ the equations of motion in TT coordinates are immediately integrated. The result is

$$p_v(u) = C_v \ , \ \ p_2(u) = C_2 \ , \ \ p_3(u) = C_3 \ , \ \ p_u = \frac{1}{2C_v} \left[m^2 - g^{\alpha\beta}(u) \ C_\alpha C_\beta \right] , \tag{43}$$

where C_2 and C_3 are constants. A particle moving in direction of the GW propagation before the infall of the latter, i.e., $p_2 = p_3 = 0$ will not change its direction. If we additionally have $C_v = \frac{mc}{\sqrt{2}}$, the particle is at rest both before and after the GW infall, since $p_1(u) = p_2(u) = p_3(u) = 0$ and $p_0 = mc$. An observer at rest, characterized by a four-velocity $V^i = \frac{1}{\sqrt{2}} (\delta_u^i + \delta_v^i) \equiv \delta_0^i$, would measure the (invariant) particle energy $\mathcal{E} \equiv p^k V_k = \frac{1}{\sqrt{2}} (p_u + p_v)$. For neutral particles p_v and p_u are given in Eq. (43). Corresponding expressions for charged particles will be obtained below.

In Fermi coordinates the situation is as follows. The system (35) reduces to the set of equations

$$\dot{y} = -C_v^{-1} p_y \ , \ \ \ \dot{z} = -C_v^{-1} p_z \ ,$$

$$C_v^{-1} \dot{p}_y = -A(u) \ y - B(u) \ z \ , \ \ \ C_v^{-1} \dot{p}_z = A(u) \ z - B(u) \ y \ , \tag{44}$$

for y, z, p_y, and p_z. If the latter set of quantities is known, the component p_u follows via (37). The quantity $\bar{v}(u)$ may be found by solving the equation

$$\dot{v} = C_v^{-1} p_u + 2\mathcal{H}(u, y, z) \ . \tag{45}$$

The set (44) is a linear, homogeneous first-order system of differential equations. For a GW field with polarization $B(u) \equiv 0$ and with A depending on the retarded time via the dimensionless variable ku, it can be written as

$$y'' - \mathcal{A}(ku) \ y = 0 \ , \ \ \ z'' + \mathcal{A}(ku) \ z = 0 \ . \tag{46}$$

Differentiation with respect to ku is denoted by a prime and $\mathcal{A}(ku) \equiv A(u)/k^2$. For $A(u) \equiv 0$, i.e. in the absence of a gravitational field, $y(u)$ and $z(u)$ are linear functions of the retarded time (and of the affine parameter τ), and the particle has constant momentum. For a periodic function $A(u)$ the equations (46) are of the type of Hill's equation [2, 7, 8]. The solutions are Hill functions. For

a dependence $\mathcal{A}(ku) = \delta + \varepsilon \cos(ku)$ where δ and ε are constants, Eqs. (46) reduce to Mathieu equations which have solutions of the type [2, 3, 8]

$$y \propto \exp\left[\mu ku\right]\phi\left(ku\right) + \exp\left[-\mu ku\right]\psi\left(ku\right) , \qquad (47)$$

where ϕ and ψ are periodic functions with the period of $\mathcal{A}(ku)$, i.e. in the present case, $\phi(ku + 2\pi) = \phi(ku)$ and $\psi(ku + 2\pi) = \psi(ku)$. The solutions consist of products of an exponential function and a periodic function of period 2π. The characteristic exponent μ is a complex constant. For $Re(\mu) = 0$ the solution is stable. In general, it is not periodic again but it is oscillating ([7], p.115). For $Re(\mu) \neq 0$ either the first or the second exponential function in (47) is unbounded and the solution is unstable. The stability properties of Mathieu's equations are well known in the literature and may be visualized by stability regions in an δ - ε diagram (see [2], Eq.(4.1) and Fig. 5.1). Let's consider the stability region which is closest to the origin $\delta = \varepsilon = 0$. For small positive values of δ and ε the boundary of this region is determined by the line $\delta = \frac{1}{4} - \frac{1}{2}\varepsilon$. Applied to the case $A(u) = A_0 \cos(ku)$, i.e. $\delta = 0$ and $\varepsilon = A_0/k^2$, this means stable solutions for $\varepsilon = \frac{A_0}{k^2} < 1/2$. Since $A_0 = \ddot{\beta}(0) = \beta_0 k^2$, we have also $\beta_0 < 1/2$. Under this condition neutral particles are parametrically oscillating in Fermi coordinates.

While equations of the type of Mathieu's equation and questions of stability will play an essential role in the following investigations of the dynamics of charged particles (see subsection 5.2.2 below), it is obvious that the description for neutral particles is more involved in Fermi coordinates. Therefore, for computational ease and in order to separate charge effects from the neutral particle motion we shall perform the following analysis in TT coordinates.

5.2 Charged particles

To obtain exactly solvable models for the particle motion we resort to simple field configurations $F_{ik}^{(A)}$. For particles with electric charge we focus on the motion of the latter in a constant homogeneous magnetic field H_0 orthogonal to the GW front plane, which corresponds to a Maxwell tensor

$$F_{jk} = H_0 \left(\delta_j^2 \delta_k^3 - \delta_j^3 \delta_k^2\right) . \qquad (48)$$

A corresponding generalization for non-Abelian fields with parallel potentials according to (18) is

$$F_{jk}^{(A)} = q^{(A)} M \left(\delta_j^2 \delta_k^3 - \delta_j^3 \delta_k^2\right), \quad M = \text{const} . \qquad (49)$$

Both (48) and (49) satisfy (39) with (31). This constitutes a model in which the gravitational wave and the fields (48) or (49) are given, external fields which are independent of each other. It may provide the basis of a perturbative treatment with respect to the GW amplitude within a linearized theory. It is worth mentioning, that the expressions (48) and (49) are also solutions of the Maxwell- and Yang-Mills equations, respectively, on the background of the exact GW (30)

(or (27)). This allows a study of the corresponding field dynamics on a GW background, which, however, is not the purpose of the present paper.

For the equations of motion in TT coordinates we obtain

$$\frac{\mathrm{d}p_2}{\mathrm{d}\tau} = \frac{M\sigma}{m}\left(g^{32}p_2 + g^{33}p_3\right) \ , \quad \frac{\mathrm{d}p_3}{\mathrm{d}\tau} = -\frac{M\sigma}{m}\left(g^{22}p_2 + g^{23}p_3\right) \ . \tag{50}$$

Equivalent second-order equations are

$$\frac{\mathrm{d}^2 p_2}{\mathrm{d}u^2} + R_2(u)\frac{\mathrm{d}p_2}{\mathrm{d}u} + W_2(u)p_2 = 0 \ , \quad p_3 = \frac{1}{g^{33}}\left(\frac{1}{\Pi}\dot{p}_2 - g^{23}p_2\right) \ , \tag{51}$$

or

$$\frac{\mathrm{d}^2 p_3}{\mathrm{d}u^2} + R_3(u)\frac{\mathrm{d}p_3}{\mathrm{d}u} + W_3(u)p_3 = 0 \ , \quad p_2 = \frac{1}{g^{22}}\left(-\frac{1}{\Pi}\dot{p}_3 - g^{23}p_3\right) \ , \tag{52}$$

where we have introduced the notations

$$\begin{aligned}
R_2(u) &= -\frac{\dot{g}^{33}(u)}{g^{33}(u)} = 2\left(\frac{\dot{L}}{L} - \dot{\beta} - \dot{\gamma}\tanh(2\gamma)\right) \ , \\
R_3(u) &= -\frac{\dot{g}^{22}(u)}{g^{22}(u)} = 2\left(\frac{\dot{L}}{L} + \dot{\beta} - \dot{\gamma}\tanh(2\gamma)\right) \ ,
\end{aligned} \tag{53}$$

and

$$\begin{aligned}
W_2(u) &= \frac{\Pi^2}{L^4} + \frac{2\Pi}{L^2}\left(\dot{\beta}\sinh(2\gamma) - \frac{\dot{\gamma}}{\cosh(2\gamma)}\right) \ , \\
W_3(u) &= \frac{\Pi^2}{L^4} + \frac{2\Pi}{L^2}\left(\dot{\beta}\sinh(2\gamma) + \frac{\dot{\gamma}}{\cosh(2\gamma)}\right) \ ,
\end{aligned} \tag{54}$$

with

$$\Pi = \frac{M\sigma}{C_v} = \mathrm{const} \ . \tag{55}$$

The substitution $\beta \rightarrow -\beta$ converts R_3 into R_2 and vice versa, while W_3 is obtained from W_2 by $\beta \rightarrow -\beta$ and simultaneously $\Pi \rightarrow -\Pi$.

By the substitution

$$p_\alpha = Z_\alpha(u)\exp\left\{-\frac{1}{2}\int_0^u R_\alpha(\zeta)d\zeta\right\} \tag{56}$$

the equations (51) and (52) may be transformed into the Hill equations

$$\ddot{Z}_\alpha + F_\alpha(u)Z_\alpha = 0 \ , \tag{57}$$

where

$$F_\alpha = W_\alpha - \frac{R_\alpha^2}{4} - \frac{\dot{R}_\alpha}{2} \ . \tag{58}$$

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The detailed form of the relations (56) and (58) is

$$p_2 = Z_2(u)\sqrt{\cosh(2\gamma)}\,\frac{e^{\beta}}{L} \;,\; p_3 = Z_3(u)\sqrt{\cosh(2\gamma)}\,\frac{e^{-\beta}}{L} \;,\; p_\alpha(0) = Z_\alpha(0) \equiv C_\alpha,$$
(59)

and

$$F_2(u) = \frac{\Pi^2}{L^4} + \frac{2\Pi}{L^2}\left(\dot{\beta}\sinh(2\gamma) - \frac{\dot{\gamma}}{\cosh(2\gamma)}\right) + \ddot{\beta} + \ddot{\gamma}\tanh(2\gamma)$$

$$+\frac{3(\dot{\gamma})^2}{(\cosh(2\gamma))^2} + (\dot{\beta})^2(\sinh(2\gamma))^2 + 2\frac{\dot{L}}{L}\dot{\beta} + 2\frac{\dot{L}}{L}\dot{\gamma}\tanh(2\gamma) - 2\dot{\gamma}\dot{\beta}\tanh(2\gamma)\;,\;\;(60)$$

$$F_3(u) = \frac{\Pi^2}{L^4} + \frac{2\Pi}{L^2}\left(\dot{\beta}\sinh(2\gamma) + \frac{\dot{\gamma}}{\cosh(2\gamma)}\right) - \ddot{\beta} + \ddot{\gamma}\tanh(2\gamma)$$

$$+\frac{3(\dot{\gamma})^2}{(\cosh(2\gamma))^2} + (\dot{\beta})^2(\sinh(2\gamma))^2 - 2\frac{\dot{L}}{L}\dot{\beta} + 2\frac{\dot{L}}{L}\dot{\gamma}\tanh(2\gamma) + 2\dot{\gamma}\dot{\beta}\tanh(2\gamma)\;,\;\;(61)$$

respectively. In a next step we have to solve Hill's equations (57).

5.2.1 General solution of Hill's equation

The structure of the solutions of the linear, second-order differential equations (57) is [2, 3, 7, 8, 9]

$$Z_2(u) = C_2\,H_2(u) - \Pi\,C_3\,H_3(u)\;.$$
(62)

The functions $H_\alpha(u)$ satisfy the initial conditions

$$H_2(0) = 1,\quad \dot{H}_2(0) = 0,\quad H_3(0) = 0,\quad \dot{H}_3(0) = 1\;,$$
(63)

and represent the fundamental solutions of Hill's equation (57) with unitary Wronsky determinant. For $Z_3(u)$ we have

$$Z_3(u) = C_3\,H_3^* + C_2\Pi\,H_2^*\;,$$
(64)

where

$$H_2^* = -\frac{L^2}{\Pi^2\cosh 2\gamma}\left[\dot{H}_2 + H_2\left(\dot{\gamma}\tanh 2\gamma + \dot{\beta} - \frac{\dot{L}}{L} - \frac{\Pi}{L^2}\sinh 2\gamma\right)\right]\;,\quad (65)$$

$$H_3^* = \frac{L^2}{\cosh 2\gamma}\left[\dot{H}_3 + H_3\left(\dot{\gamma}\tanh 2\gamma + \dot{\beta} - \frac{\dot{L}}{L} - \frac{\Pi}{L^2}\sinh 2\gamma\right)\right]\;,\quad (66)$$

$$H_2^*(0) = \dot{H}_2(0) = 0,\quad H_3^*(0) = \dot{H}_3(0) = 1\;.$$
(67)

In the absence of gravitational radiation, i.e. for $\beta = \gamma \equiv 0$, $L \equiv 1$, the functions F_α in equation (58) reduce to

$$F_2(u) = F_3(u) = \text{const} = \Pi^2\;.$$
(68)

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Equation (57) then describes harmonic oscillations with

$$H_2 = H_3^* \equiv \cos \Pi u, \quad H_3 = H_2^* \equiv \frac{1}{\Pi} \sin \Pi u \ . \tag{69}$$

Since with (42) and (55) we have $\Pi u \to \Omega_H \tau$ where $\Omega_H \equiv \frac{eH_0}{mc}$ is the Larmor frequency, we recover the corresponding particle rotation in flat spacetime. Generally, the functions H_2, H_3, H_2^* and H_3^* cannot be written in terms of elementary functions but are given as series representations. In the following subsection we shall be interested in expressions for F_2 and F_3 for which the Hill equations (57) specify to Mathieu equations. Then the functions H_α can be expanded in powers of the GW amplitude, where the zeroth order is given by (69).

The analysis so far may be summarized by writing the solution of the equations of motion (51) and (52) in the compact and elegant matrix form

$$\begin{pmatrix} p_2 \\ p_3 \end{pmatrix} = \mathbf{H}(u) \cdot \begin{pmatrix} C_2 \\ C_3 \end{pmatrix} \ , \tag{70}$$

where

$$\mathbf{H}(u) \equiv \frac{\sqrt{\cosh 2\gamma}}{L} \begin{pmatrix} e^\beta & 0 \\ 0 & e^{-\beta} \end{pmatrix} \cdot \begin{pmatrix} H_2(u) & -\Pi H_3(u) \\ \Pi H_2^*(u) & H_3^*(u) \end{pmatrix} \ . \tag{71}$$

The set of equations (70) and (71) represents the general solution for the momentum of the charged particle in the GW field (30) and the Yang-Mills field (49).

5.2.2   A simple model

Now we apply the general formalism to a "sandwich" GW (see, e.g., [33]) with polarization $\gamma = 0$. Let β be different from zero during a finite retarded time interval T, i.e., $\beta = 0$ for $u \leq 0$ and $u \geq T$. Within the interval $0 < u < T$ we assume β to be periodic according to

$$\beta(u) = \beta_0(1 - \cos(ku)) \ , \qquad (0 < u < T) \ . \tag{72}$$

This implies $\beta(0) = \beta(2\pi/k) = 0$ and $\dot{\beta}(0) = \dot{\beta}(2\pi/k) = 0$. Furthermore, the time scale T is assumed to be small compared with the scale on which the background factor L changes [33]. Under these conditions we may neglect the $\dot{L}/L$ terms in (60) and (61) and use the latter expressions with $L = 1$. For this situation the potentials F_2 and F_3 reduce to

$$\begin{aligned} F_2 &= \Pi^2 + \beta_0 k^2 \cos(ku) = \Pi^2 - R^2_{.u2u} \ , \\ F_3 &= \Pi^2 - \beta_0 k^2 \cos(ku) = \Pi^2 - R^3_{.u3u} \end{aligned} \tag{73}$$

in the interval $0 < u < T$. Replacing now the variable u by ku and denoting the derivative with respect to ku again by a prime, the equations (57) with (73)

specify to

$$Z_2'' + \left(\frac{\Pi^2}{k^2} + \beta_0 \cos(ku)\right) Z_2 = 0 \ ,$$

$$Z_3'' + \left(\frac{\Pi^2}{k^2} - \beta_0 \cos(ku)\right) Z_3 = 0 \ . \tag{74}$$

Both Z_2 and Z_3 obey Mathieu equations. In the absence of the GW, i.e. for $\beta \equiv 0$, we have $F_\alpha(ku) = \Pi^2/k^2 = \text{const}$ (here we have used the redefinition $F_\alpha(ku) \equiv F_\alpha(u)/k^2$) and the equations of motion reduce to harmonic oscillator equations with solutions

$$Z_2 = p_2 = C_2 \cos(\Pi u) - C_3 \sin(\Pi u) \tag{75}$$

and

$$Z_3 = p_3 = C_3 \cos(\Pi u) + C_2 \sin(\Pi u) \ . \tag{76}$$

Replacing here Π and u according to (55) and (42), we find a particle rotation in the $x^2 0 x^3$ plane with the angular velocity $\Omega_M \equiv \frac{M\sigma}{m}$, which is, of course, the analogue of the Larmor frequency. Immediately after the wave front, i.e. at $u = 0_{+0}$, we have

$$F_2(0) = \frac{\Pi^2}{k^2} + \beta_0 \ , \quad F_3(0) = \frac{\Pi^2}{k^2} - \beta_0 \ . \tag{77}$$

The jump $\beta_0 k^2$ of the curvature tensor at the front makes the evolutions of the p_2 and p_3 components different. They begin to oscillate with different frequencies and the particle trajectory is no longer circular. Corresponding features hold for the second polarization $\beta \equiv 0$ and $\gamma \neq 0$, for which F_2 and F_3 differ in the term linear in Π [cf. Eqs. (60) and (61)].

The general solutions of Eq. (74) are of the type of "cosine elliptic" and "sine elliptic" functions (see [7, 9]). As already mentioned, the latter may be expanded in powers of β_0 with the zeroth-order terms (75) and (76).

Eqs. (74) are of the same type as Eqs. (46). With the identifications $\delta \rightarrow \Pi^2/k^2$ and $\varepsilon \rightarrow \beta_0$, the stability discussion mentioned in subsection 5.1 may be applied here as well (see [2], Eq.(4.1) and Fig. 5.1). Depending on the parameter combinations the solutions may be stable or unstable. Within the stable regions the functions Z_2 and Z_3 are parametrically oscillating which, according to (59) implies a corresponding behavior of the particle momenta. The regions of stable solutions are characterized by stability zones in a $(\Pi/k)^2 \times \beta_0$ plane which are connected together at the points $(\Pi/k)^2 = n^2/4$, $\beta_0 = 0$, where n is an integer [2].

The analysis of the neutral particle motion in subsection 5.1 corresponds to the case $\Pi = n = 0$. For $\Pi \neq 0$ the relevant values for the transition points are $n = 1, 2, ...$ For $n = 1$ we have $\Pi/k = 1/2$. These points on the axis $\beta_0 = 0$ (which corresponds to the absence of the GW) are the only transition points between stable regions which also belong to the stable region. All other

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boundary points of the stable regions are unstable points. Consequently, any deviation from $\beta_0 = 0$, i.e. even a GW with arbitrary weak amplitude β_0, induces an instability in these critical points. In particular, this is true for the point $\Pi/k = 1/2$ (see [2], Fig. 5.1). This demonstrates that parametric instabilities are a generic phenomenon for the motion of particles in all the cases considered here. While we have obtained this result in TT coordinates, the transformations (34) allow us to find the corresponding particle momenta in Fermi coordinates as well. The relevant transformations are

$$p_{\bar{v}} = p_v = C_v \,, \quad p_y = p_2(u)\frac{e^{-\beta}}{L} - C_v \left(\frac{\dot{L}}{L} + \dot{\beta} \right) y(u) \,,$$

$$p_z = p_3(u)\frac{e^{\beta}}{L} - C_v \left(\frac{\dot{L}}{L} - \dot{\beta} \right) z(u) \,, \tag{78}$$

where

$$y(u) = Le^{\beta} \left[y(0) - \int_0^u d\zeta p_2(\zeta) L^{-2}(\zeta) e^{-2\beta(\zeta)} \right] \,,$$

$$z(u) = Le^{-\beta} \left[z(0) - \int_0^u d\zeta p_3(\zeta) L^{-2}(\zeta) e^{2\beta(\zeta)} \right] \,. \tag{79}$$

It is interesting to realize that there are astrophysical situation for which the existence of such kind of instabilities might be relevant. This can be seen with the help of the following order-of-magnitude estimates. Eq. (55) may be written as $\Pi = \Omega_M m/C_v$ with $\Omega_M = M\sigma/m$. For the electromagnetic case one has $\sigma = e$ and $M = H_0$. The interstellar magnetic field is of the order $3 \div 6 \cdot 10^{-6}$Oe [34]. The integral of motion C_v is equal to $C_v = (\sqrt{m^2 + \vec{p}^2(0)} - p^1(0))/\sqrt{2}$. For nonrelativistic particles $C_v \propto m/\sqrt{2}$ and $\Pi \propto \omega_H \sqrt{2}$. (The coefficient $\sqrt{2}$ disappears if we use the natural parameter τ instead of retarded time u). Using the estimate [34] $\omega_H \propto 10^7 \cdot H_0$(Hz) we find $\omega_H \approx 10^1 \div 10^2$ Hz. This is well within the typical range $1 - 10^3$Hz for the frequency k of a GW, generated by rapidly rotating neutron stars (pulsars). Thus, for non-relativistic particles Π may be of the same order as the GW frequency k. The situation is different for ultrarelativistic particles. Since for particles that move in propagation direction of the GW ($p^1(0) > 0$), we have $C_v \to 0$ for $m \to 0$. Consequently, the quantity Π becomes very large [cf. Eq. (55)], such that $\Pi >> k$. However, if the particles move in the opposite direction, i.e. $p^1(0) < 0$, any value of C_v is possible. In particular, Π is not excluded to be in the range of about 10^{-3}Hz which is typical for infra-low-frequency GW from relativistic compact binaries.

Since an ensemble of particles will generally not be characterized by a single value of C_v but by a distribution, the quantity C_v may play the role of a tuning parameter in the following sense. Let us associate a mean value $\langle C_v \rangle$ for the system as a whole and let the system be outside the resonance $\omega_H/k = n/2$ for this $\langle C_v \rangle$, but not very far from it. Since $\Pi = \omega_H m/C_v$, we will very likely find a particular particle with a specific C_v such that for this particle $\Pi/k = n/2$ exactly. Consequently, certain particles of the ensemble might be resonant under these conditions.

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6 Evolution of the spin four-vector

In this section we focus on the solution of Eq. (5) in external gravitational and magnetic fields. The equations (4) and (5) admit two integrals of motion, namely $p_i S^i = 0$ and $S_i S^i = -E_0^2 = \text{const}$ [20]. In the present context this amounts to

$$p_v S_u + p_u S_v + p_\alpha S^\alpha = 0, \quad 2 S_u S_v + S_\alpha S^\alpha = -E_0^2 \; , \tag{80}$$

which may be used to eliminate the components S_v and S_u according to

$$S_u = \frac{1}{2 p_v} \left[-p_\alpha S^\alpha \mp \sqrt{(p_\alpha S^\alpha)^2 + (m^2 - p_\alpha p^\alpha)(E_0^2 + S_\alpha S^\alpha)} \right] \; , \tag{81}$$

and

$$S_v = \frac{1}{2 p_u} \left[-p_\alpha S^\alpha \pm \sqrt{(p_\alpha S^\alpha)^2 + (m^2 - p_\alpha p^\alpha)(E_0^2 + S_\alpha S^\alpha)} \right] \; , \tag{82}$$

respectively. After the reparametrization (42) the remaining equations are

$$\frac{dS_\alpha}{du} = \frac{1}{2} g^{\sigma\rho} \dot{g}_{\rho\alpha} \left(S_\sigma - p_\sigma \frac{S_v}{C_v} \right) + \frac{e}{2 C_v} \left[g F_{\alpha k} \, S^k + \frac{(g-2)}{m^2} p_\alpha F_{kl} S^k p^l \right] \; . \tag{83}$$

Furthermore, the equation for S_v is

$$\frac{dS_v}{du} = \frac{e(g-2)}{2 m^2} F_{kl} S^k p^l. \tag{84}$$

In the following we shall restrict ourselves to the exactly integrable case $g = 2$. Under this condition we find from (84) that

$$S_v = \text{const} = E_v \; . \tag{85}$$

After the substitution

$$S_\alpha = X_\alpha + \frac{E_v}{C_v} p_\alpha \; , \tag{86}$$

where p_α is assumed to be given by Eq. (70) in terms of Hill (or Mathieu) functions, we obtain a homogeneous equation for the new variable X_α,

$$\frac{\mathrm{d} X_\alpha}{\mathrm{d} u} = \frac{1}{2} g^{\sigma\rho} \dot{g}_{\rho\alpha} X_\sigma + \frac{e}{C_v} F_{\alpha\sigma} g^{\sigma\lambda} \, X_\lambda \; . \tag{87}$$

It is convenient to write this equation in the matrix form

$$\frac{\mathrm{d}}{\mathrm{d} u} \mathbf{X} = \left(\mathbf{A} + \frac{\Pi}{L^2} \mathbf{B} \right) \cdot \mathbf{X} \; , \tag{88}$$

where $\mathbf{X}$ is a column vector with elements X_2 and X_3. The two-dimensional matrices $\mathbf{A}$ and $\mathbf{B}$ have the structures

$$\mathbf{A} = \begin{pmatrix} \frac{\dot{L}}{L} + \cosh^2(2\gamma)\dot{\beta} & e^{2\beta}(\dot{\gamma} - \sinh(2\gamma)\cosh(2\gamma)\dot{\beta}) \\ e^{-2\beta}(\dot{\gamma} + \sinh(2\gamma)\cosh(2\gamma)\dot{\beta}) & \frac{\dot{L}}{L} - \cosh^2(2\gamma)\dot{\beta} \end{pmatrix} \; , \tag{89}$$

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and

$$\mathbf{B} \equiv \begin{pmatrix} \sinh 2\gamma & -e^{2\beta}\cosh 2\gamma \\ e^{-2\beta}\cosh 2\gamma & -\sinh 2\gamma \end{pmatrix} , \tag{90}$$

respectively. Equations (50) may be written in matrix form as well:

$$\begin{pmatrix} p_2 \\ p_3 \end{pmatrix}^{\cdot} = \frac{\Pi}{L^2}\,\mathbf{B} \cdot \begin{pmatrix} p_2 \\ p_3 \end{pmatrix} . \tag{91}$$

For the derivative of the combination $p_2^2 + p_3^2$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(p_2^2 + p_3^2\right) = \frac{2\Pi}{L^2}\left[\left(p_2^2 - p_3^2\right)\sinh 2\gamma - 2p_2 p_3 \cosh 2\gamma \sinh 2\beta\right] . \tag{92}$$

In general, the right-hand side of this equation is different from zero, i.e., the particle motion is no longer circular in the GW field.

Eq. (88) may be further simplified by changing to a new variable $\mathbf{Y}$, defined by

$$\mathbf{X} = \mathbf{T} \cdot \mathbf{Y} , \tag{93}$$

where $\mathbf{T}$ is supposed to satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{T} = \mathbf{A} \cdot \mathbf{T} . \tag{94}$$

This procedure [cf. [35, 36]] removes the $\mathbf{A}$ term in Eq. (88)and gives rise to the equation

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{Y} = \frac{\Pi}{L^2}\,\hat{\mathbf{B}} \cdot \mathbf{Y} \tag{95}$$

for $\mathbf{Y}$ with

$$\hat{\mathbf{B}} \equiv \mathbf{T}^{-1} \cdot \mathbf{B} \cdot \mathbf{T} . \tag{96}$$

By direct calculation one checks that the matrix

$$\mathbf{T} = L \cdot \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix} \cdot \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \cdot \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} , \tag{97}$$

where

$$\psi \equiv \int_0^u \dot{\beta}\sinh 2\gamma\,\mathrm{d}u , \tag{98}$$

satisfies the equation (94). The determinants of each of the three two-dimensional matrices in (97) are equal to one. In the absence of the GW field all of them are identical to $\mathbf{I}$, i.e.,

$$\mathbf{T}(0) = \mathbf{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{99}$$

The structure of the third matrix on the right-hand side of Eq. (97) suggests the interpretation as a gravitationally induced rotation with phase $\psi(u)$ and

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frequency $\dot{\psi}(u)$. For either of the polarizations $\gamma = 0$ or $\beta = 0$, however, we have $\psi = 0$ and the third matrix reduces to $\mathbf{I}$.

Direct calculation of the matrix $\hat{\mathbf{B}}$ in (96) with the help of the expressions (90), (97) and (98) yields the surprisingly simple result

$$\hat{\mathbf{B}} \equiv \mathbf{T}^{-1} \cdot \mathbf{B} \cdot \mathbf{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{100}$$

It is remarkable, that the matrix $\hat{\mathbf{B}}$, different from $\mathbf{B}$, does *not* depend on retarded time. This property allows us to find the solution of equation (95) in terms of elementary functions as

$$\begin{pmatrix} Y_2(u) \\ Y_3(u) \end{pmatrix} = \mathbf{R}(u) \cdot \begin{pmatrix} Y_2(0) \\ Y_3(0) \end{pmatrix}, \quad \mathbf{R}(u) \equiv \begin{pmatrix} \cos \Phi(u) & -\sin \Phi(u) \\ \sin \Phi(u) & \cos \Phi(u) \end{pmatrix}, \tag{101}$$

where

$$\Phi(u) \equiv \Pi \int_0^u \frac{du}{L^2(u)} . \tag{102}$$

The combination
$$Y_2^2(u) + Y_3^2(u) = Y_2^2(0) + Y_3^2(0) \tag{103}$$

is preserved, i.e. the dynamics of $\mathbf{Y}$ represents a rotation in the $x^2 0 x^3$ plane. The functions $S_2(u)$ and $S_3(u)$ in (86)) can now be expressed in terms of the three matrices $\mathbf{H}(u)$, $\mathbf{T}(u)$ and $\mathbf{R}(u)$, given by the expressions (71), (97) and (101), respectively:

$$\begin{pmatrix} S_2(u) \\ S_3(u) \end{pmatrix} = \mathbf{T}(u) \cdot \mathbf{R}(u) \cdot \begin{pmatrix} S_2(0) \\ S_3(0) \end{pmatrix} + \frac{E_v}{C_v} \left[\mathbf{H}(u) - \mathbf{T}(u) \cdot \mathbf{R}(u) \right] \begin{pmatrix} C_2 \\ C_3 \end{pmatrix} . \tag{104}$$

While the matrices $\mathbf{T}(u)$ and $\mathbf{R}(u)$ are constructed out of elementary functions, the matrix $\mathbf{H}(u)$, according to Eq. (71), consists of Hill functions, which for the special case of subsection 5.2.2 reduce to Mathieu functions. The latter, in turn, can be expressed via "cosine elliptic" and "sine elliptic" functions (see the discussion following Eqs. (77)). All these functions are assumed to be known here. In the absence of the GW,

$$\mathbf{T}(u) \equiv \mathbf{I}, \quad \mathbf{H}(u) = \mathbf{R}(u) \equiv \mathbf{R}_0(\tau) = \begin{pmatrix} \cos \Omega_H \tau & -\sin \Omega_H \tau \\ \sin \Omega_H \tau & \cos \Omega_H \tau \end{pmatrix}, \tag{105}$$

and we recover the standard flat spacetime rotation of the spin particle,

$$\begin{pmatrix} p_2(\tau) \\ p_3(\tau) \end{pmatrix} = \mathbf{R}_0(\tau) \cdot \begin{pmatrix} C_2 \\ C_3 \end{pmatrix}, \quad \begin{pmatrix} S_2(\tau) \\ S_3(\tau) \end{pmatrix} = \mathbf{R}_0(\tau) \cdot \begin{pmatrix} S_2(0) \\ S_3(0) \end{pmatrix}. \tag{106}$$

Equation (104) represents the general solution for the spin dynamics in the GW field (30) and the magnetic field (48). The structure of the solution (104) allows us to interpret the spin dynamics as composed of three separate contributions,

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characterized by the matrices $\mathbf{R}(u)$, $\mathbf{T}(u)$ and $\mathbf{H}(u)$. The matrix $\mathbf{R}$ represents a Larmor type precession with the frequency $\frac{\Pi}{L^2(u)}$. As already mentioned, the matrix $\mathbf{T}$ describes a gravitationally induced rotation with phase $\psi(u)$ and frequency $\dot{\psi}(u)$, and finally, the matrix $\mathbf{H}$ accounts for the coupling of the particle motion (70) to the spin dynamics.

7 Isospin evolution

In this section as well as in the subsequent one we discuss the dynamics of non-Abelian charges under the influence of external gravitational and Yang-Mills fields. Let us consider Eq. (20) for the $SU(2)$ symmetry group. Since the structure constants for this group coincide with the Levi-Civita symbol, all three directions in the isospin space are equivalent. With the choice $I^{(3)} = I_{(A)}q^{(A)}$ we obtain the following equations for the isospin evolution:

$$\frac{\mathrm{d}I^{(1)}}{\mathrm{d}\tau} = \Omega \cdot I^{(2)} \ , \quad \frac{\mathrm{d}I^{(2)}}{\mathrm{d}\tau} = -\Omega \cdot I^{(1)} \ . \tag{107}$$

The precession frequency Ω in Eq. (20) is calculated on the particle worldline with p^i from (70) and the potential

$$A_i(u){=}\frac{1}{2}M\left[\left(x^2(u){-}x^2(0){-}\frac{C_2}{\Pi C_v}\right)\delta_i^3 - \left(x^3(u){-}x^3(0){-}\frac{C_3}{\Pi C_v}\right)\delta_i^2\right] \ , \tag{108}$$

corresponding to the constant solution $F_{23} = M$. Differentiating the expression (108), we recover the field strength (49). The frequency $\Omega(u)$ in (20) is given by

$$\Omega(u) = \frac{g}{m}(A_2 p^2 + A_3 p^3) \ . \tag{109}$$

Here, the arbitrary constant was chosen such that $\Omega(u = 0) = 0$. In order to find the terms $x^2(u){-}x^2(0)$ and $x^3(u){-}x^3(0)$ which are needed in (108), we have to integrate the second equation in (35). The formal solution is

$$x^\alpha(u){-}x^\alpha(0) = \frac{1}{C_v}\int_0^u d\xi g^{\alpha\beta}(\xi)p_\beta(\xi) \ , \tag{110}$$

which provides us with

$$\Omega(u) = \frac{gM}{2mC_v}\left\{\int_0^u \mathrm{d}\xi \left[p^2(\xi)p^3(u) - p^3(\xi)p^2(u)\right] - \frac{1}{\Pi}\left[C_2 p^3(u) - C_3 p^2(u)\right]\right\} \ . \tag{111}$$

Again we assume here $p^2(u)$ and $p^3(u)$ to be known, i.e., the particle dynamics is considered to be solved [cf, Eqs. (70) and (71)]. The solution of the system (107) then becomes

$$I^{(1)} = I\cos\Psi(u) \ , \quad I^{(2)} = -I\sin\Psi(u) \ , \quad \Psi(u) = \Psi(0) + \frac{m}{C_v}\int_0^u \Omega(u)\mathrm{d}u \ . \tag{112}$$

20

The function $\Psi(u)$ plays the role of the (generally u-dependent) phase of the isospin precession in the external Yang-Mills field [37]. The set of equations (112) with (111) provides a complete description for the isospin dynamics under the influence of the GW (30) and the Yang-Mills field (49).

8   Color charge evolution

The SU(3) case may be studied along similar lines although it is technically more extended since more degrees of freedom are involved. As a result, we shall find a richer dynamical structure than in the SU(2) case. With the ansatz (18) and the expressions (15), Eqs. (24) for the color charge dynamics become

$$
\begin{aligned}
\frac{\mathrm{d}Q^{(1)}}{\mathrm{d}\tau} &= -\Omega\left[(q^{(2)}Q^{(3)}-q^{(3)}Q^{(2)})+\frac{1}{2}(q^{(4)}Q^{(7)}-q^{(7)}Q^{(4)})-\frac{1}{2}(q^{(5)}Q^{(6)}-q^{(6)}Q^{(5)})\right] , \\
\frac{\mathrm{d}Q^{(2)}}{\mathrm{d}\tau} &= -\Omega\left[(q^{(3)}Q^{(1)}-q^{(1)}Q^{(3)})+\frac{1}{2}(q^{(4)}Q^{(6)}-q^{(6)}Q^{(4)})+\frac{1}{2}(q^{(5)}Q^{(7)}-q^{(7)}Q^{(5)})\right] , \\
\frac{\mathrm{d}Q^{(3)}}{\mathrm{d}\tau} &= -\Omega\left[(q^{(1)}Q^{(2)}-q^{(2)}Q^{(1)})+\frac{1}{2}(q^{(4)}Q^{(5)}-q^{(5)}Q^{(4)})-\frac{1}{2}(q^{(6)}Q^{(7)}-q^{(7)}Q^{(6)})\right] , \\
\frac{\mathrm{d}Q^{(4)}}{\mathrm{d}\tau} &= -\frac{\Omega}{2}\left[(q^{(7)}Q^{(1)}-q^{(1)}Q^{(7)})+(q^{(6)}Q^{(2)}-q^{(2)}Q^{(6)})+(q^{(5)}Q^{(3)}-q^{(3)}Q^{(5)})\right] \\
&\quad -\frac{\sqrt{3}\Omega}{2}[q^{(5)}Q^{(8)}-q^{(8)}Q^{(5)}] , \\
\frac{\mathrm{d}Q^{(5)}}{\mathrm{d}\tau} &= -\frac{\Omega}{2}\left[(q^{(1)}Q^{(6)}-q^{(6)}Q^{(1)})+(q^{(7)}Q^{(2)}-q^{(2)}Q^{(7)})+(q^{(3)}Q^{(4)}-q^{(4)}Q^{(3)})\right] \\
&\quad -\frac{\sqrt{3}\Omega}{2}[q^{(8)}Q^{(4)}-q^{(4)}Q^{(8)}] , \\
\frac{\mathrm{d}Q^{(6)}}{\mathrm{d}\tau} &= -\frac{\Omega}{2}\left[(q^{(5)}Q^{(1)}-q^{(1)}Q^{(5)})+(q^{(2)}Q^{(4)}-q^{(4)}Q^{(2)})+(q^{(3)}Q^{(7)}-q^{(7)}Q^{(3)})\right] \\
&\quad -\frac{\sqrt{3}\Omega}{2}[q^{(7)}Q^{(8)}-q^{(8)}Q^{(7)}] , \\
\frac{\mathrm{d}Q^{(7)}}{\mathrm{d}\tau} &= -\frac{\Omega}{2}\left[(q^{(1)}Q^{(4)}-q^{(4)}Q^{(1)})+(q^{(2)}Q^{(5)}-q^{(5)}Q^{(2)})+(q^{(6)}Q^{(3)}-q^{(3)}Q^{(6)})\right] \\
&\quad -\frac{\sqrt{3}\Omega}{2}[q^{(8)}Q^{(6)}-q^{(6)}Q^{(8)}] , \\
\frac{\mathrm{d}Q^{(8)}}{\mathrm{d}\tau} &= -\frac{\sqrt{3}\Omega}{2}\left[(q^{(4)}Q^{(5)}-q^{(5)}Q^{(4)})+(q^{(6)}Q^{(7)}-q^{(7)}Q^{(6)})\right] . \quad (113)
\end{aligned}
$$

The space of color charges may be split into three different subspaces, which correspond to the structures of the $SU(2)$, $SU(2)\times U(1)$ and $U(1)$ subgroups of the total group $SU(3)$ (see, e.g. [38]). In the following subsections we consider the vector $q^{(A)}$ to lie in the first, second, and third subspaces, respectively.

8.1 First special case

Let the vector $q^{(A)}$ have only the three non-zero components $q^{(1)}$, $q^{(2)}$ and $q^{(3)}$. It is then evident that

$$(Q^{(1)})^2 + (Q^{(2)})^2 + (Q^{(3)})^2 = \text{const} , \tag{114}$$

$$(Q^{(4)})^2 + (Q^{(5)})^2 + (Q^{(7)})^2 + (Q^{(7)})^2 = \text{const} , \tag{115}$$

and

$$Q^{(8)} = \text{const} . \tag{116}$$

For the color charges $Q^{(1)}$, $Q^{(2)}$, and $Q^{(3)}$, which correspond to a $SU(2)$ subgroup of the total $SU(3)$ group, the combination (114) is preserved. A similar relation holds for the set $Q^{(4)}$, $Q^{(5)}$, $Q^{(6)}$, and $Q^{(7)}$, while $Q^{(8)}$ is separately conserved. Relations (114)-(116) are also obtained for the case that the only non-vanishing components are $q^{(4)}$, $q^{(5)}$, $q^{(6)}$, and $q^{(7)}$, as well as for the choice $q^{(1)} = q^{(2)} = ... = q^{(7)} = 0$ and $q^{(8)} \neq 0$. Let us now further specify to the case $q^{(A)} = \delta_{(1)}^{(A)}$. Then the system (113) takes the form

$$\frac{\mathrm{d}Q^{(1)}}{\mathrm{d}\tau} = 0 , \quad \frac{\mathrm{d}Q^{(8)}}{\mathrm{d}\tau} = 0 ,$$

$$\frac{\mathrm{d}Q^{(2)}}{\mathrm{d}\tau} = \Omega Q^{(3)} , \qquad \frac{\mathrm{d}Q^{(3)}}{\mathrm{d}\tau} = -\Omega Q^{(2)} ,$$

$$\frac{\mathrm{d}Q^{(4)}}{\mathrm{d}\tau} = \frac{1}{2}\Omega Q^{(7)} , \qquad \frac{\mathrm{d}Q^{(7)}}{\mathrm{d}\tau} = -\frac{1}{2}\Omega Q^{(4)} ,$$

$$\frac{\mathrm{d}Q^{(5)}}{\mathrm{d}\tau} = -\frac{1}{2}\Omega Q^{(6)} , \qquad \frac{\mathrm{d}Q^{(6)}}{\mathrm{d}\tau} = \frac{1}{2}\Omega Q^{(5)} . \tag{117}$$

The color charge $Q^{(1)}$ remains constant because it is the projection $Q^{(A)}q_{(A)}$ of the vector $Q^{(A)}$ on the given preferred direction $q_{(A)}$. The charge $Q^{(8)}$ does not evolve, because for such a $q_{(A)}$ the antisymmetric tensor $H_{(A)(B)}$ in (24) does not contain a non-vanishing component with $(A) = (8)$. The equations for $Q^{(2)}$, ... $Q^{(7)}$ split into *three two-dimensional* subsystems with the pairs $Q^{(2)}$ and $Q^{(3)}$, $Q^{(4)}$ and $Q^{(7)}$, $Q^{(5)}$ and $Q^{(6)}$. The evolution of the first pair ($Q^{(2)}$ and $Q^{(3)}$) corresponds to a precession in the group space with the frequency Ω. It has a solution of the type (112). The dynamics of the pairs $Q^{(4)}$, $Q^{(7)}$ and $Q^{(5)}$, $Q^{(6)}$ is a precession with the frequency $\Omega/2$.

8.2 Second special case: $q^{(A)} = \delta_{(4)}^{(A)}$

Now we assume $q^{(A)}$ to lie in the second subspace. As an example we consider the case $q^{(A)} = \delta_{(4)}^{(A)}$. For this choice the set of equations (113) can be transformed into

$$\frac{\mathrm{d}Q^{(4)}}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\frac{\sqrt{3}}{2}Q^3 + \frac{1}{2}Q^8\right) = 0 ,$$

$$\frac{\mathrm{d}Q^{(5)}}{\mathrm{d}\tau} = \Omega Q^* \ , \quad \frac{\mathrm{d}Q^*}{\mathrm{d}\tau} = -\Omega Q^{(5)} \ , \quad Q^* \equiv \frac{1}{2}(Q^{(3)} + \sqrt{3}Q^{(8)}) \ ,$$

$$\begin{aligned}
\frac{\mathrm{d}Q^{(1)}}{\mathrm{d}\tau} &= -\frac{1}{2}\Omega Q^{(7)} \ , & \frac{\mathrm{d}Q^{(7)}}{\mathrm{d}\tau} &= \frac{1}{2}\Omega Q^{(1)} \ , \\
\frac{\mathrm{d}Q^{(2)}}{\mathrm{d}\tau} &= -\frac{1}{2}\Omega Q^{(6)}, & \frac{\mathrm{d}Q^{(6)}}{\mathrm{d}\tau} &= \frac{1}{2}\Omega Q^{(2)} \ .
\end{aligned} \tag{118}$$

The quantities $Q^{(4)}$ and $-\frac{\sqrt{3}}{2}Q^3 + \frac{1}{2}Q^8$ remain constant, the projections $Q^{(5)}$ and $Q^* \equiv \frac{1}{2}(Q^{(3)} + \sqrt{3}Q^{(8)})$ precess with the frequency Ω, the pairs $Q^{(1)}$, $Q^{(7)}$ and $Q^{(2)}$, $Q^{(6)}$ precess with $\Omega/2$. Furthermore, one has

$$(Q^{(1)})^2 + (Q^{(7)})^2 = \mathrm{const} \ , \quad (Q^{(2)})^2 + (Q^{(6)})^2 = \mathrm{const} \ ,$$

$$(Q^{(3)})^2 + (Q^{(5)})^2 + (Q^{(8)})^2 = \mathrm{const} \ . \tag{119}$$

8.3 Third special case: $q^{(A)} = \delta^{(A)}_{(8)}$

Finally, let $q^{(A)}$ be parallel to the basis vector in the $U(1)$ subspace, i.e., $q^{(A)} = \delta^{(A)}_{(8)}$. Here we obtain

$$\frac{\mathrm{d}Q^{(1)}}{\mathrm{d}\tau} = 0 \ , \quad \frac{\mathrm{d}Q^{(2)}}{\mathrm{d}\tau} = 0 \ , \quad \frac{\mathrm{d}Q^{(3)}}{\mathrm{d}\tau} = 0 \ , \quad \frac{\mathrm{d}Q^{(8)}}{\mathrm{d}\tau} = 0 \ ,$$

as well as

$$\begin{aligned}
\frac{\mathrm{d}Q^{(4)}}{\mathrm{d}\tau} &= \frac{\sqrt{3}}{2}\Omega Q^{(5)} \ , & \frac{\mathrm{d}Q^{(5)}}{\mathrm{d}\tau} &= -\frac{\sqrt{3}}{2}\Omega Q^{(4)} \ , \\
\frac{\mathrm{d}Q^{(6)}}{\mathrm{d}\tau} &= \frac{\sqrt{3}}{2}\Omega Q^{(7)} \ , & \frac{\mathrm{d}Q^{(7)}}{\mathrm{d}\tau} &= -\frac{\sqrt{3}}{2}\Omega Q^{(6)} \ .
\end{aligned} \tag{120}$$

The pairs $Q^{(4)}$, $Q^{(5)}$ and $Q^{(6)}$, $Q^{(7)}$ precess with the frequency $\frac{\sqrt{3}}{2}\Omega$, while $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$, and $Q^{(8)}$ are constant.

8.4 Remarks on the general case

In the general case one expects the color vector $Q^{(A)}$ to rotate in the hypersurface orthogonal to $q^{(A)}$ in the group space. This is illustrated by the following analogy. Let us consider the standard decomposition of the Maxwell tensor with respect to a four-velocity vector V^i of an arbitrary observer,

$$F_{ik} = E_i V_k - E_k V_i - \varepsilon_{ikjl} H^j V^l \ , \tag{121}$$

where E^i and H^k are the corresponding four-vectors for the electric and magnetic fields, respectively. For $F_{ik}V^k = 0$ the comoving observer experiences a magnetic field only which, via the Lorentz force generates a spatial particle rotation, i.e. a rotation in the hypersurface orthogonal to V^i.

In the present case the vector $q^{(A)}$ plays the role of V^i. Instead of Maxwell's tensor we have to consider [cf. Eq. (24)] the antisymmetric tensor $H_{(A)(C)} = f_{(A)(B)(C)}q^{(B)}$. Since $H_{(A)(C)}q^{(C)} = 0$, this represents a group space analogue to the previous case of a pure magnetic field in spacetime. Consequently, the color vector rotates in the group space analogously to the momentum vector of a charged particle in a pure magnetic field.

9   Conclusions

Charged particles in electromagnetic fields are known to be parametrically influenced by gravitational waves. Typical phenomena are parametric resonances and parametric oscillations. In the present paper we have generalized and extended work in this field to include classical spin particles and particles with non-Abelian charges in specific Yang-Mills fields. Moreover, no weak field approximation was used for the GW. The electrically charged spin particle was described by the Bargmann-Michel-Telegdi equations. For the dynamics of the non-Abelian charges we used Wong's equations for the isospin (SU(2)-symmetry) and for the color charges (SU(3)-symmetry). We derived exact general solutions for the parametric influence of the GW on the particle motion in each of the mentioned cases, including the dynamics of spin, isospin, and color charge. For the case of a special sandwich GW the particle dynamics was reduced to a set of Mathieu equations. Using well-known stability properties of the latter we found that parametric instabilities are a generic phenomenon for such kind of particle motion. Since spin, isospin and color charge are coupled to this motion, their dynamics is affected correspondingly. The spin dynamics was shown to be composed of three elements, namely a gravitationally modified Larmor precession, a part due to the coupling to the particle motion, and a pure gravitational part. The vectors of isospin and color charge carry out gravitationally influenced precession motions in their group spaces which we have classified for several cases.

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